

Quasi-degenerate Neutrino mass models and their significance: A model independent investigation

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Abstract

The prediction of possible ordering of neutrino masses relies mostly on the model selected. Alienating the $\mu - \tau$ interchange symmetry from discrete flavour symmetry based models, turns the neutrino mass matrix less predictive. But this inspires one to seek the answer from other phenomenological frameworks. We need a proper parametrization of the neutrino mass matrices concerning individual hierarchies. In the present work, we attempt to study the six different cases of Quasi-degenerate (QDN) neutrino models. The related mass matrices, m_{LL}^ν are parametrized with two free parameters (α, η) and standard Wolfenstein parameter (λ). The input mass scale m_0 is selected around $\sim 0.08 \text{ eV}$. We begin with a $\mu - \tau$ symmetric neutrino mass matrix tailed by a correction from charged lepton sector. The parametrization accentuates the existence of four independent texture zero building block matrices which are common to all the QDN models under $\mu - \tau$ symmetric framework. These remain invariant irrespective of any choice of solar angle. In our parametrization, the neutrino sector controls the solar angle, whereas the reactor and atmospheric angles are dictated by the charged lepton sector. In the framework of oscillation experiments, cosmological observation and future experiments involving β -decay and $0\nu\beta\beta$ experiments, all QDN models are tested and a reason to rule out anyone out of the six models is unfounded. A strong preference for $\sin^2 \theta_{12} = 0.32$ is observed for QDNH-*Type A* model.

1 Introduction

One of the most challenging riddles of neutrino physics is to trace out the exact ordering of the absolute neutrino masses. The Quasi degenerate hierarchy [1–14] among all the three possibilities, refers to the scenario when the three mass eigenvalues are of similar order, $m_1 \sim m_2 \sim m_3$. As the solar mass squared difference (Δm_{21}^2) is positive and the the sign of atmospheric mass squared difference (Δm_{31}^2) is unspecified, we encounter two divisions of QDN patterns: they are,

- “Quasi-degenerate Normal Hierarchy (QDNH) type” : $m_1 \lesssim m_2 \lesssim m_3$,
- “Quasi-degenerate Inverted Hierarchy type” (QDIH): $m_3 \lesssim m_1 \lesssim m_2$.

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Besides, the remaining possibilities are strict “Normal hierarchy” (NH): $m_1 \ll m_2 \ll m_3$, $m_1 \sim 0$ and “Inverted hierarchy” (IH): $m_3 \ll m_1 \ll m_2$, $m_3 \sim 0$. The two Majorana phases (α, β) are admitted to the diagonalized neutrino mass matrix m_{LL}^d , where, $m_{LL}^d = \text{diag}(m_1, m_2 e^{i\alpha}, m_3 e^{i\beta})$ [15]. On adopting the CP conserving cases, three subclasses corresponding to each model is generated. The CP parity patterns of the sub classes are :

- Type IA: $m_{LL}^d = \text{diag}(+m_1, -m_2, +m_3)$,
- Type IB: $m_{LL}^d = \text{diag}(+m_1, +m_2, +m_3)$ and,
- Type IC: $m_{LL}^d = \text{diag}(+m_1, +m_2, -m_3)$.

The QDN model were very often forsaken [16, 17] in view of the neutrino-less double β decay experiments and cosmological data. The range of absolute neutrino mass scale, m_0 was chosen as, $0.1 \text{ eV} - 0.4 \text{ eV}$ [18] in earlier QDN models. But, the Cosmological data in concern with the sum of the three absolute neutrino masses, $\Sigma|m_i| \leq 0.28 \text{ eV}$ [19], strongly abandons any possibility of quasi-degenerate neutrinos to exist with absolute mass scale more than 0.1 eV . The Σm_i corresponding to strict NH and IH scenarios are approximately 0.06 eV and 0.1 eV respectively. Hence the validity of both the models are beyond dispute. In the context of cosmological observation on Σm_i and the future experiments, we shall try to look into the possibilities related to the reanimation of the QDN models, with comparatively lower mass scale, $m_0 \lesssim 0.1 \text{ eV}$.

Regarding the three unknown absolute masses, only two relations involving $m_{i=1,2,3}^2$ are known so far. In NH and IH models, this problem can be easily overcome as the lowest mass (either m_1 or m_2) is set to zero. In case of QDN model, we consider the largest mass m_0 as a input. Besides, there are also three mixing angles: reactor (θ_{13}), solar (θ_{12}) and atmospheric (θ_{23}). A general neutrino mass matrix m_{LL}^ν carries the information of all these six quantities. For a phenomenological analysis of the QDN model, a suitable parametrization of m_{LL}^ν is an essential part. We shall try to design the general neutrino mass matrix, m_{LL}^ν with minimum numbers of free parameters. As a first approximation, m_{LL}^ν is assumed to follow $\mu - \tau$ symmetry [20–25]. This symmetry keeps θ_{12} arbitrary and hence can handle both Tri-Bimaximal(TBM) mixing and deviation from it as well [12–14, 26, 27]. This characteristic feature of $\mu - \tau$ symmetry bears immense phenomenological importance. The expected deviations to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$, are controlled from charged lepton sector [28–38].

We hope, this investigation on QDN mass models will serve as a platform for our future study of Baryogenesis and leptogenesis [12–14, 39]. This investigation will require the knowledge of the texture of left handed neutrino mass matrices, m_{LL}^ν .

2 Need for parametrization of a general $\mu - \tau$ symmetric mass matrix.

The present neutrino oscillation data reports the lepton mixing angles, $\theta_{23} \sim 40^\circ$ and $\theta_{13} \sim 9^\circ$ [40–46] which are undoubtedly deviated from what TBM mixing and BM mixing says: $\theta_{23} = 45^\circ$ and $\theta_{13} = 0^\circ$. A neutrino mass matrix which satisfies these properties (of BM/TBM mixing) [20, 21, 47–52], in the basis where charged lepton mass matrix, m_{LL}^l is diagonal, $m_{LL}^l = \text{diag}(m_e, m_\mu, m_\tau)$, exhibits a $\mu - \tau$ interchange symmetry. With a permutation matrix, T which conducts a flavor interchange $\mu \leftrightarrow \tau$, we express the texture of a $\mu - \tau$ symmetric mass matrix as in the following [53, 54],

$$T M_{\mu\tau} T = M_{\mu\tau}, \implies M_{\mu\tau} = \begin{bmatrix} x & y & y \\ y & z & w \\ y & w & z \end{bmatrix}, \quad (1)$$

where,

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (2)$$

We experience another form $M_{\mu\tau}$ [55], different from that in Eq.(1),

$$M_{\mu\tau} = \begin{bmatrix} x & y & -y \\ y & z & w \\ -y & w & z \end{bmatrix}. \quad (3)$$

The matrix element is invariant under the flavor interchange of $\mu \leftrightarrow -\tau$. The permutation matrix responsible for this symmetry is T' .

$$T' M_{\mu\tau} T' = M_{\mu\tau}, \implies M_{\mu\tau}, \quad (4)$$

where,

$$T' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \quad (5)$$

The presence of a $-ve$ sign before y in the 1-3 the element of $M_{\mu\tau}$ (see Eq.(3)) ensures the positivity of the mixing angles.

Except the maximal atmospheric and vanishing reactor angle, $\mu-\tau$ symmetry has no further prediction. Different discrete symmetry groups are very often combined with $\mu-\tau$ interchange symmetry in order to obtain a predictive neutrino mass matrix [56]. For example, in the original Altarelli-Feruglio model [?, 57], the neutrino mass matrix takes the form (see Eq.(1)),

$$M_{\mu\tau} = \begin{bmatrix} a + \frac{2}{3}b & -\frac{b}{3} & -\frac{b}{3} \\ -\frac{b}{3} & \frac{2}{3}b & a - \frac{b}{3} \\ -\frac{b}{3} & a - \frac{b}{3} & \frac{2}{3}b \end{bmatrix}, \quad (6)$$

the mass eigenvalues are $m_1 = a + b$, $m_2 = a$ and $m_3 = b - a$ which gives, $\Delta m_{sol}^2 = (-b^2 - 2ab)$ and $\Delta m_{atm}^2 = -4ab$. Since it is known that $\Delta m_{sol}^2 > 0$, which implies $ab < 0$. Hence, a and b must have opposite signs which in turn says, $\Delta m_{atm}^2 > 0$. This model advocates for normal hierarchy of the absolute neutrino masses. In addition, it supports for TBM mixing: $\theta_{12} = \sin^{-1}(1/\sqrt{3})$. Similarly, a neutrino mass matrix of following kind (see Eq.(1)),

$$M_{\mu\tau} = \begin{bmatrix} 0 & a & a \\ a & b & c \\ a & c & b \end{bmatrix}, \quad (7)$$

can be related with an interesting mixing scheme called Golden ratio [58] and dictates the mass pattern to be of normal hierarchy type.

One of the most interesting property of the $\mu - \tau$ (interchange) symmetry which is very often neglected is the arbitrariness of the solar angle θ_{12} . With a proper choice of the parameters, x, y, w and z , θ_{12} is controlled with the following relation [22], intrinsic to $M_{\mu\tau}$ in Eq.(1),

$$\tan 2\theta_{12} = \frac{2\sqrt{2}y}{x - w - z}. \quad (8)$$

For, $M_{\mu\tau}$ in Eq.(3) the expression for $\tan 2\theta_{12}$ is,

$$\tan 2\theta_{12} = \frac{2\sqrt{2}y}{x + w - z} \quad (9)$$

It seems that $\mu - \tau$ symmetry is more natural and BM and TBM mixing schemes are certain special cases of this symmetry. In fact, the recent result $\sin^2 \theta_{12} \sim 0.32$ [44] deviated a little from TBM prediction ($\sin^2 \theta_{12} = 0.33$), can be accommodated within the $\mu - \tau$ symmetry regime [12–14, 26, 27]. Neglecting the small deviations (though significant) for θ_{23} and θ_{13} , we first approximate, $m_{LL}^\nu = M_{\mu\tau}$ and the charged lepton diagonalizing matrix, $U_{eL} = I$.

This is an undeniable fact that the predictions of a suitable order of absolute neutrino masses are not unique and differs with the choice of models. Keeping aside all the models, which associate $M_{\mu\tau}$ with different discrete flavour symmetries, here we concentrate on a general parametrization of $M_{\mu\tau}$. The idea behind this decoupling is not to overlook the necessity of different symmetry groups, but to look into the subtle aspects of $\mu - \tau$ symmetry, starting from a phenomenological point of view. Here we emphasize on the facts that $\mu - \tau$ interchange symmetry is not partial to any hierarchy of absolute neutrino masses and has a good control over the solar angle. In the Refs. [59, 60], we put forward another possible way to parametrize the neutrino mass matrix based on $\mu - \tau$ symmetry.

In the present article, concerning the parametrization of the $\mu - \tau$ symmetric mass matrices for different hierarchical cases, we shall stick to the second convention (see Eq.(3)).

3 Invariant building blocks of $\mu - \tau$ symmetric mass matrix

We want to draw attention on the general texture of $M_{\mu\tau}$ s satisfying BM [47–49] and TBM [20, 21, 50–52] mixing schemes.

$$M_{\mu\tau}^{BM} = \begin{bmatrix} x & y & y \\ y & z & x - z \\ y & x - z & z \end{bmatrix}, \quad (10)$$

$$M_{\mu\tau}^{TBM} = \begin{bmatrix} x & y & y \\ y & z & x + y - z \\ y & x + y - z & z \end{bmatrix}. \quad (11)$$

It is to be noted that the above two forms of $M_{\mu\tau}$'s are in accordance with the first convention (see Eq.(1)). In terms of the three parameters x, y and z , the respective mass matrices can be decomposed with certain building block matrices $I_x, I_y^{BM, TBM}$ and I_z in the following way.

$$M_{\mu\tau}^{BM} = xI_x + yI_y^{BM} + zI_z, \quad (12)$$

$$M_{\mu\tau}^{TBM} = xI_x + yI_y^{TBM} + zI_z. \quad (13)$$

Where,

$$I_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, I_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad (14)$$

$$I_y^{BM} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, I_y^{TBM} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad (15)$$

There is a distinct change in the texture of I_y , as the mixing pattern transits from BM to TBM. I_y^{BM} and I_y^{TBM} have the diagonalizing matrices, $U_{BM} = R_{23}(\theta_{23} = -\pi/4).R_{13}(\theta_{13} = 0).R_{12}(\theta_{12} = -\pi/4)$ and $U_{TBM} = R_{23}(\theta_{23} = -\pi/4).R_{13}(\theta_{13} = 0).R_{12}(\theta_{12} = \sin^{-1}(1/\sqrt{3}))$ respectively and thus carry the signatures of respective models. For, $I_{x,z}$ the diagonalizing matrices are, $U_{x,y} = R_{23}(\theta_{23} = -\pi/4)$.

We insist on the possibility of finding out certain building blocks of $M_{\mu\tau}$ that will remain invariant at the face of any mixing schemes (BM or TBM) or simply independent of any θ_{12} in general. With this idea, four such independent texture-zero matrices, $I_{i=0,1,2,3}$ are posited (see Table.(1)). On considering the fact that a general $M_{\mu\tau}$ is capable of holding four free parameters at the most (if α and β are specified), we parametrize $M_{\mu\tau}$ for QDNH-*Type IA* case in the following way.

$$M_{\mu\tau} = I_0 - (\beta - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1 - 2\eta^2)^{1/2}I_3. \quad (16)$$

$$= \begin{pmatrix} \alpha - \beta - 2\alpha\eta^2 & -\alpha\eta(1 - 2\eta^2)^{1/2} & \alpha\eta(1 - 2\eta^2)^{1/2} \\ -\alpha\eta(1 - 2\eta^2)^{1/2} & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{1/2} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 \end{pmatrix} \quad (17)$$

Here, α , β and η are three free parameters and the mass matrix is normalized with input parameter m_0 . The parameters, α and β are related with absolute masses of three neutrinos. The quantity, m_0 signifies the largest neutrino mass. It can be seen that whatever may be the changes in mixing schemes, the basic building blocks are not affected. The free parameter η dictates the solar angle. $\eta = 1/2, 1/\sqrt{6}$, correspond to BM and TBM mixing respectively. In contrast to $M_{\mu\tau}$ s in Eqs. (12)-(13), the corresponding mass matrices are,

$$M_{\mu\tau}^{BM} = I_0 - (\beta - \frac{\alpha}{2})I_1 + 0I_2 + \frac{1}{2\sqrt{2}}\alpha I_3, \quad (18)$$

$$M_{\mu\tau}^{TBM} = I_0 - (\beta - \frac{\alpha}{2})I_1 - \frac{1}{6}\alpha I_2 + \frac{1}{3}\alpha I_3. \quad (19)$$

Here we want to add that with $\eta = 2/5$, $\sin^2 \theta_{12} = 0.32$ (best-fit) [44] can be obtained. It can be seen that, $I_0 + I_1 = I$, the identity matrix. Also, from Table.(1), this is interesting to note that the diagonalizing matrix of I_3 is none other than U_{BM} .

There are certain significant features of this parametrization. With same building block matrices, we can extend the parametrization of $M_{\mu\tau}$ s for other five QDN and even for the NH and IH cases also. For example, similar to Eq. (16), a rearrangement of the free parameters (α, β, η) , and I_i s we parametrize $M_{\mu\tau}$ for QDIH-*Type IA* case in the following way.

$$M_{\mu\tau} = \beta I_0 - \left(1 - \frac{\alpha}{2}\right) I_1 + 2\alpha \left(\eta^2 - \frac{1}{4}\right) I_2 + \alpha\eta(1 - 2\eta^2)^{1/2} I_3. \quad (20)$$

Upon considering $\beta = \alpha$, in Eq.(16), we get $M_{\mu\tau}$ satisfying strict NH-*Type IA* condition.

$$M_{\mu\tau} = I_0 - \frac{\alpha}{2}I_1 + 2\alpha \left(\eta^2 - \frac{1}{4}\right) I_2 + \alpha\eta(1 - 2\eta^2)^{1/2} I_3. \quad (21)$$

I_i		I_i^{diag}	U_i
I_0	$\frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
I_1	$\frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
I_2	$\frac{1}{2} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
I_3	$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Table 1: The texture of the invariant building blocks $I_{i=0,1,2,3}$, the diagonalized blocks $I_{i=0,1,2,3}^{diag}$ and the corresponding diagonalizing matrices (U_i).

Similarly, with $\beta = 0$, in Eq.(20), we obtain a $M_{\mu\tau}$ that represents IH-*Type IA* case.

$$M_{\mu\tau} = 0I_0 - \left(1 - \frac{\alpha}{2}\right) I_1 + 2\alpha \left(\eta^2 - \frac{1}{4}\right) I_2 + \alpha\eta (1 - 2\eta^2)^{1/2} I_3. \quad (22)$$

Similarly, we can formulate the same for other cases also. The details are shown in Table.(2) and Table.(3)

In this present approach, the mass parameters and the mixing angle parameters are decoupled. A single expression of $\tan 2\theta_{12}$ for all the eleven cases is,

$$\tan 2\theta_{12} = \frac{2\sqrt{2}\eta(1 - 2\eta^2)^{1/2}}{1 - 4\eta^2}, \quad (23)$$

$$\text{or, } \sin^2 \theta_{12} = 2\eta^2. \quad (24)$$

4 The input parameter m_0 for QDN model

In either of the two QDN cases, m_0 represents the largest absolute neutrino mass. For QDNH cases, we use the following relations to work out the neutrino masses m_i .

$$m_1 = m_0 \sqrt{1 - \frac{\Delta m_{atm}^2}{m_0^2}}, \quad (25)$$

$$m_2 = m_0 \sqrt{1 + \frac{\Delta m_{sol}^2}{m_0^2} - \frac{\Delta m_{atm}^2}{m_0^2}}, \quad (26)$$

$$m_3 = m_0. \quad (27)$$

QDN-NH,IH	$M_{\mu\tau}(\alpha, \beta, \eta)/m_0$	m_i/m_0
QDNH-IA :	$\begin{bmatrix} \alpha - \beta - 2\alpha\eta^2 & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 \end{bmatrix}$ $= I_0 - (\beta - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{array}{c} \alpha - \beta \\ -\beta \\ 1 \end{array}$
QDNH-IB :	$\begin{bmatrix} \beta + 2\alpha\eta^2 - \alpha & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \frac{\beta}{2} + \alpha\eta^2 & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 \end{bmatrix}$ $= I_0 + (\beta - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3$	$\begin{array}{c} \beta - \alpha \\ \beta \\ 1 \end{array}$
QDNH-IC :	$\begin{bmatrix} \beta + 2\alpha\eta^2 - \alpha & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{\beta}{2} - \alpha\eta^2 - \frac{1}{2} & \alpha\eta^2 - \frac{1}{2} - \frac{\beta}{2} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \frac{1}{2} - \frac{\beta}{2} & \frac{\beta}{2} - \alpha\eta^2 - \frac{1}{2} \end{bmatrix}$ $= -I_0 + (\beta - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3$	$\begin{array}{c} \beta - \alpha \\ \beta \\ -1 \end{array}$
QDIH-IA :	$\begin{bmatrix} \alpha - 2\alpha\eta^2 - 1 & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{\beta}{2} + \alpha\eta^2 - \frac{1}{2} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 & \frac{\beta}{2} + \alpha\eta^2 - \frac{1}{2} \end{bmatrix}$ $= \beta I_0 - (1 - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{array}{c} \alpha - 1 \\ -1 \\ \beta \end{array}$
QDIH-IB :	$\begin{bmatrix} 1 - \alpha + 2\alpha\eta^2 & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 & \frac{\beta}{2} + \alpha\eta^2 - \frac{1}{2} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{\beta}{2} + \alpha\eta^2 - \frac{1}{2} & \frac{1}{2} + \frac{\beta}{2} - \alpha\eta^2 \end{bmatrix}$ $= \beta I_0 + (1 - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{array}{c} 1 - \alpha \\ 1 \\ \beta \end{array}$
QDIH-IC :	$\begin{bmatrix} 1 - \alpha + 2\alpha\eta^2 & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \frac{\beta}{2} - \alpha\eta^2 & \alpha\eta^2 - \frac{\beta}{2} - \frac{1}{2} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \frac{\beta}{2} - \frac{1}{2} & \frac{1}{2} - \frac{\beta}{2} - \alpha\eta^2 \end{bmatrix}$ $= -\beta I_0 + (1 - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{array}{c} 1 - \alpha \\ 1 \\ -\beta \end{array}$

Table 2: The parametrization of $M_{\mu\tau}$ for six different QDN cases with three free parameters (α, β, η) with four basic building blocks $I_{i=0,1,2,3}$. m_0 is the input parameter.

NH,IH	$M_{\mu\tau}(\alpha, \eta)/m_0$	m_i/m_0
NH- IA :	$\begin{bmatrix} -2\alpha\eta^2 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \frac{\alpha}{2} + \alpha\eta^2 & \frac{1}{2} + \frac{\alpha}{2} - \alpha\eta^2 \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\alpha}{2} - \alpha\eta^2 & \frac{1}{2} - \frac{\alpha}{2} + \alpha\eta^2 \end{bmatrix}$ $= I_0 - \frac{\alpha}{2}I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\begin{matrix} 0 \\ -\alpha \\ 1 \end{matrix}$
NH- IB :	$\begin{bmatrix} 2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \frac{\alpha}{2} - \alpha\eta^2 & \frac{1}{2} - \frac{\alpha}{2} + \alpha\eta^2 \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \frac{\alpha}{2} + \alpha\eta^2 & \frac{1}{2} + \frac{\alpha}{2} - \alpha\eta^2 \end{bmatrix}$ $= I_0 + \frac{\alpha}{2}I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1-2\eta^2)^{1/2}I_3$	$\begin{matrix} 0 \\ \alpha \\ 1 \end{matrix}$
NH- IC :	$\begin{bmatrix} 2\alpha\eta^2 & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{\alpha}{2} - \alpha\eta^2 - \frac{1}{2} & \alpha\eta^2 - \frac{\alpha}{2} - \frac{1}{2} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \frac{\alpha}{2} - \frac{1}{2} & \frac{\alpha}{2} - \alpha\eta^2 - \frac{1}{2} \end{bmatrix}$ $= -I_0 - \frac{\alpha}{2}I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\begin{matrix} 0 \\ \alpha \\ -1 \end{matrix}$
IH- IA :	$\begin{bmatrix} \alpha - 2\alpha\eta^2 - 1 & -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta(1-2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \frac{1}{2} & \frac{1}{2} - \alpha\eta^2 \\ -\alpha\eta(1-2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \alpha\eta^2 & \alpha\eta^2 - \frac{1}{2} \end{bmatrix}$ $= 0I_0 - (1 - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\begin{matrix} \alpha - 1 \\ -1 \\ 0 \end{matrix}$
IH- IB :	$\begin{bmatrix} 1 - \alpha + 2\alpha\eta^2 & \alpha\eta(1-\eta^2)^{\frac{1}{2}} & -\alpha\eta(1-\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1-\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \alpha\eta^2 & \alpha\eta^2 - \frac{1}{2} \\ -\alpha\eta(1-\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \frac{1}{2} & \frac{1}{2} - \alpha\eta^2 \end{bmatrix}$ $= 0I_0 - (1 - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1-2\eta^2)^{1/2}I_3.$	$\begin{matrix} 1 - \alpha \\ 1 \\ 0 \end{matrix}$

Table 3: The extension of the parametrization to NH and IH models. It can be seen that only two free parameters α and η are required to parametrize the mass matrices.

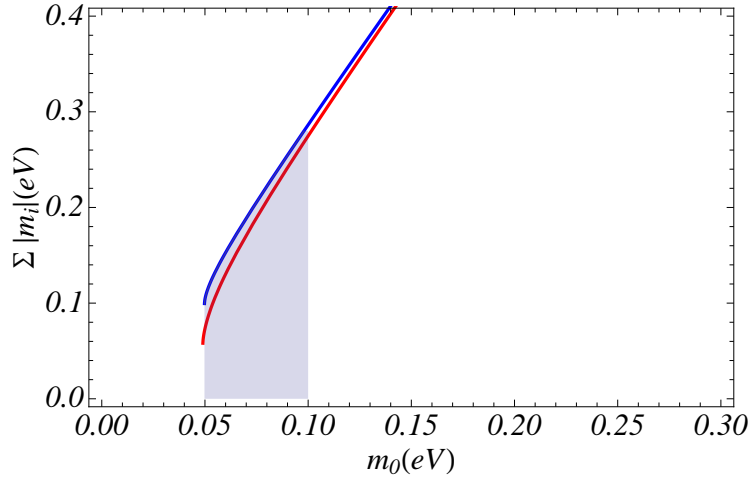


Figure 1: Σm_i vs the input parameter m_0 . Corresponding to the cosmological upper bound $\Sigma m_i \lesssim 0.28 \text{ eV}$ and beyond $m_0 > 0.05 \text{ eV}$, Σm_i is imaginary, we get a range of m_0 as $[0.05, 0.1] \text{ eV}$. The Red stands for QDNH case while Blue signifies the case of QDIH

For, QDIH cases, we use,

$$m_1 = m_0 \sqrt{1 - \frac{\Delta m_{sol}^2}{m_0^2}}, \quad (28)$$

$$m_2 = m_0, \quad (29)$$

$$m_3 = m_0 \sqrt{1 - \frac{\Delta m_{sol}^2}{m_0^2} - \frac{\Delta m_{atm}^2}{m_0^2}}. \quad (30)$$

Also, we have,

$$\Sigma m_i = |m_1| + |m_2| + |m_3|. \quad (31)$$

The present cosmological upper bound on Σm_i is 0.28 eV [19] and the best-fit values of the mass squared differences are approximately: $\Delta m_{21}^2 \sim 7.6 \times 10^{-5} \text{ eV}^2$, $\Delta m_{31}^2 \sim 2.4 \times 10^{-3} \text{ eV}^2$ [44–46]. From a graphical analysis of $\Sigma |m_i|$ vs. m_0 reveals that the absolute mass scale m_0 must lie approximately within $0.05 \text{ eV} - 0.1 \text{ eV}$ (Fig.(1)). The upper limit of m_0 is the direct outcome of the cosmological upper bound [19]. The lower limit arises because, when $m_0 \lesssim 0.05 \text{ eV}$, m_1 , m_2 for QDNH case and m_3 for QDIH case become imaginary. By studying the variation of m_i and corresponding slopes (dm_i/dm_0) with respect to m_0 (Fig.(2)), we expect that the level of degeneracy is better for $m_0 > 0.07 \text{ eV}$ and approximate the range of m_0 from $0.07 - 0.1 \text{ eV}$. For all numerical studies we adhere to $m_0 \sim 0.08 \text{ eV}$.

5 Endeavor to suppress the number of free parameters in QDN models

This is clear that only two free parameters α and η are required to parametrize $M_{\mu\tau}$ for NH and IH models (see Table.(3)); whereas QDN model requires three (α, β, η) (see Table.(2)). The rejection of one parameter for NH and IH cases is natural. But we shall try to see whether under certain logical ground we can suppress the number of free parameters for QDN model or not.

We consider the example of QDNH-Type IA case. With $m_0 \sim 0.08 \text{ eV}$, we study the ratio $\alpha : \beta$ and $\beta : \eta$ for the 2σ and 3σ ranges of the three parameters based on the Global data

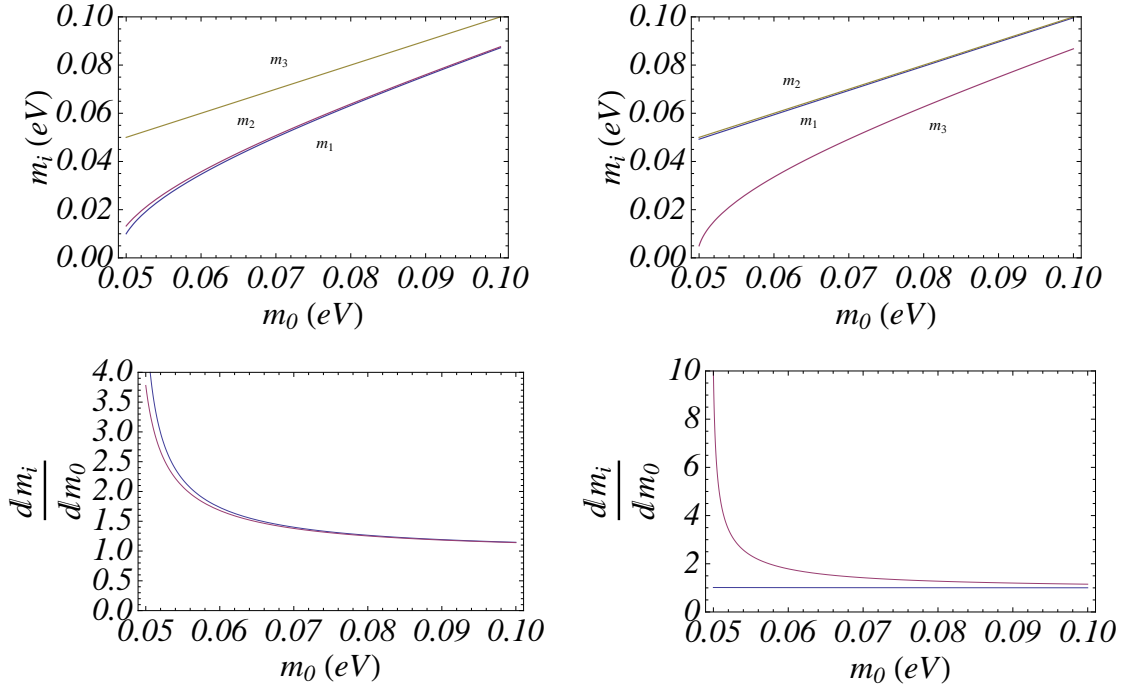


Figure 2: Study of m_i vs m_0 (top-left: QDNH case, top-right: QDIH case) and dm_i/dm_0 vs m_0 (bottom-left: QDNH case, bottom-right: QDIH case).

analysis [45]. The idea behind this approach is to detect whether there exists a simple linear correlation between the parameters or not.

Fig (3) reveals such a quest is not absurd at all and we can assume, $\alpha \cong 2\beta$ and $\beta \cong 2\eta$. But the first ansatz leads to $\Delta m_{21}^2 = 0$ and turns out insignificant. We stick to the second ansatz. An immediate outcome is that the parameter η which is responsible only for the mixing angle θ_{12} in the earlier parametrization of $M_{\mu\tau}(\alpha, \beta, \eta)$ is now capable of driving the mass parameters also. In other words, the arbitrariness of θ_{12} is now reduced a little. In contrast to Eq.(16), for

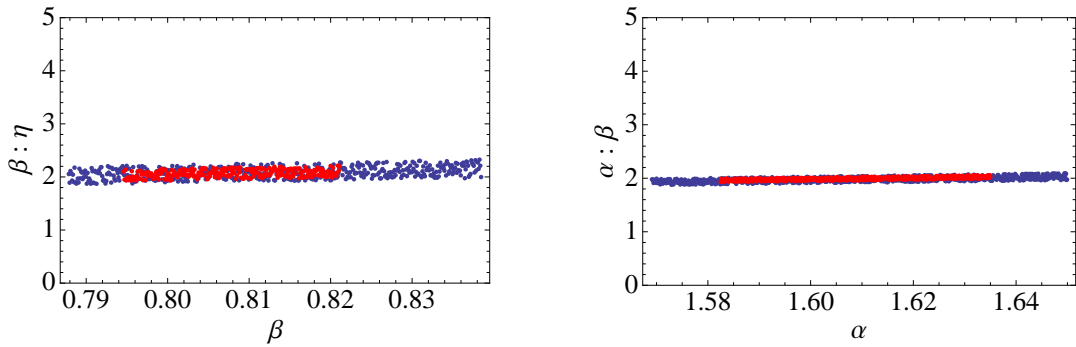


Figure 3: In order to check the validity of the assumptions: there may lie a linear correlation between the parameters (α, β, η) for QDNH-IA case, we check graphically $\alpha : \beta$ and $\beta : \eta$. The analysis hints for $\beta = 2\eta$ and $\alpha = 2\beta$. The colour red and blue stand for the 2σ and the 3σ range respectively.

QDNH-*Type A* case, with normalized m_0 , we have,

$$M_{\mu\tau}(\alpha, \eta) = I_0 - (2\eta - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1 - 2\eta^2)^{1/2}I_3, \quad (32)$$

$$= \begin{bmatrix} \alpha - 2\eta - 2\alpha\eta^2 & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \eta + \alpha\eta^2 & \frac{1}{2} + \eta - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \frac{1}{2} - \eta + \alpha\eta^2 \end{bmatrix}. \quad (33)$$

The ansatz $\beta = 2\eta$ is applicable to other remaining QDN cases also (see Table.(4)). Needless to say, that the suppression of parameters does not affect $\tan 2\theta_{12}$ in Eq.(23).

6 TBM, deviation from TBM and BM mixing

We experience that $M_{\mu\tau}$ parametrized with (α, η) (see Table.(4)) gives certain correlation between absolute masses and θ_{12} ,

$$\sin \theta_{12} = \frac{1}{\sqrt{2}} \frac{m_2}{m_3} \text{ (QDNH case)}, \quad (34)$$

$$\sin \theta_{12} = \frac{1}{\sqrt{2}} \frac{m_3}{m_2} \text{ (QDIH case)}. \quad (35)$$

Considering QDNH case as an example, we find,

$$\Delta m_{21}^2 = \alpha(2\sqrt{2}\sin \theta_{12} - \alpha)m_0^2, \quad (36)$$

$$\Delta m_{31}^2 = (1 - \alpha + \sqrt{2}\sin \theta_{12})(1 + \alpha - \sqrt{2}\sin \theta_{12})m_0^2. \quad (37)$$

For all the QDNH cases, we fix the input, $m_0 = 0.082 \text{ eV}$. TBM condition implies $\theta_{12} = \sin^{-1}(1/\sqrt{3})$. A choice of the free parameter, $\alpha = 1.626$ (QDNH-*IA*), We obtain $\Delta m_{21}^2 \sim 7.6 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 \sim 2.32 \times 10^{-3} \text{ eV}^2$ [44–46].

If we expect a little deviation from TBM mixing, say $\sin^2 \theta_{12} = 0.32$ then along with a choice of $\alpha = 1.5929$, we obtain $\Delta m_{21}^2 \sim 7.6 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 \sim 2.49 \times 10^{-3} \text{ eV}^2$ [44]. Similar treatment holds good for the remaining cases also. The parametrization of the mass matrix with two free parameters (α, η) is compatible with both TBM mixing and with deviation from TBM as well, and agrees to the global data [44–46].

But the BM mixing ($\sin \theta_{12} = 1/\sqrt{2}$) is somehow disfavoured by all the six QDN mass models $M_{\mu\tau}(\alpha, \eta)$ (see Table.(4)). The BM mixing will lead to, $m_2 = m_3$, which implies $\Delta m_{21}^2 = \Delta m_{31}^2$. (See Eqs. (34)-(35)) Needless to mention that this is problem never arises if we adopt the general parametrization with three free parameters (α, β, η) (see Table.(2)).

7 Charged lepton correction

We derive the diagonalizing matrix for both $M_{\mu\tau}(\alpha, \beta, \eta)$ (see Table. (2)) and $M_{\mu\tau}(\alpha, \eta)$ (see Table.(4)) in the exact form as shown below,

$$U_{\nu L} = \begin{pmatrix} (1 - 2\eta^2)^{1/2} & \sqrt{2}\eta & 0 \\ -\eta & \frac{1}{\sqrt{2}}(1 - 2\eta^2)^{1/2} & \frac{1}{\sqrt{2}} \\ \eta & -\frac{1}{\sqrt{2}}(1 - 2\eta^2)^{1/2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (38)$$

Indeed, θ_{13} is zero and θ_{23} is $\pi/4$. We have to include some extra ingredient in order to deviate θ_{13} and θ_{23} from what $M_{\mu\tau}$ says.

QDN-NH,IH	$M_{\mu\tau}(\alpha, \eta)/m_0$	m_i/m_0
QDNH- <i>IA</i> :	$\begin{bmatrix} \alpha - 2\eta - 2\alpha\eta^2 & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \eta + \alpha\eta^2 & \frac{1}{2} + \eta - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \frac{1}{2} - \eta + \alpha\eta^2 \end{bmatrix}$ $= I_0 - (2\eta - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{matrix} \alpha - 2\eta \\ -2\eta \\ 1 \end{matrix}$
QDNH- <i>IB</i> :	$\begin{bmatrix} 2\eta + 2\alpha\eta^2 - \alpha & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \frac{1}{2} - \eta + \alpha\eta^2 \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \eta + \alpha\eta^2 & \frac{1}{2} + \eta - \alpha\eta^2 \end{bmatrix}$ $= I_0 + (2\eta - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3$	$\begin{matrix} 2\eta - \alpha \\ 2\eta \\ 1 \end{matrix}$
QDNH- <i>IC</i> :	$\begin{bmatrix} 2\eta + 2\alpha\eta^2 - \alpha & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \eta - \alpha\eta^2 - \frac{1}{2} & \alpha\eta^2 - \frac{1}{2} - \eta \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \frac{1}{2} - \eta & \eta - \alpha\eta^2 - \frac{1}{2} \end{bmatrix}$ $= -I_0 + (2\eta - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3$	$\begin{matrix} 2\eta - \alpha \\ 2\eta \\ -1 \end{matrix}$
QDIH- <i>IA</i> :	$\begin{bmatrix} \alpha - 2\alpha\eta^2 - 1 & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \eta + \alpha\eta^2 - \frac{1}{2} & \frac{1}{2} + \eta - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \eta + \alpha\eta^2 - \frac{1}{2} \end{bmatrix}$ $= 2\eta I_0 - (1 - \frac{\alpha}{2})I_1 + 2\alpha(\eta^2 - \frac{1}{4})I_2 + \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{matrix} \alpha - 1 \\ -1 \\ 2\eta \end{matrix}$
QDIH- <i>IB</i> :	$\begin{bmatrix} 1 - \alpha + 2\alpha\eta^2 & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \eta + \alpha\eta^2 - \frac{1}{2} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \eta + \alpha\eta^2 - \frac{1}{2} & \frac{1}{2} + \eta - \alpha\eta^2 \end{bmatrix}$ $= 2\eta I_0 + (1 - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{matrix} 1 - \alpha \\ 1 \\ 2\eta \end{matrix}$
QDIH- <i>IC</i> :	$\begin{bmatrix} 1 - \alpha + 2\alpha\eta^2 & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \eta - \alpha\eta^2 & \alpha\eta^2 - \eta - \frac{1}{2} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta^2 - \eta - \frac{1}{2} & \frac{1}{2} - \eta - \alpha\eta^2 \end{bmatrix}$ $= -2\eta I_0 + (1 - \frac{\alpha}{2})I_1 - 2\alpha(\eta^2 - \frac{1}{4})I_2 - \alpha\eta(1 - 2\eta^2)^{1/2}I_3.$	$\begin{matrix} 1 - \alpha \\ 1 \\ -2\eta \end{matrix}$

Table 4: The parametrization of $M_{\mu\tau}$ for six different QDN cases with two free parameters (α, η) with four basic building blocks $I_{i=0,1,2,3}$. m_0 is the input parameter.

The mixing matrix in the lepton sector, U_{PMNS} , appears in the electro-weak coupling to the W bosons and is expressed in terms of lepton mass eigenstates. We have,

$$\mathcal{L} = -\bar{e}_L M_e e_R - \frac{1}{2} \bar{\nu}_L m_{LL}^\nu \nu_L^c + H.c., \quad (39)$$

A transformation from flavour to mass basis: $U_{eL}^\dagger M_e U_{eR} = \text{diag}(m_e, m_\mu, m_\tau)$ and $U_{\nu L}^\dagger m_{LL}^\nu U_{\nu R} = \text{diag}(m_1, m_2, m_3)$ gives [28–34],

$$U_{PMNS} = U_{eL}^\dagger U_{\nu L}. \quad (40)$$

As stated earlier, it was assumed $U_{eL} = I$ and hence $U_{PMNS} = U_{\nu L}(\eta)$. Probably a suitable texture of U_{eL} other than I , satisfying the unitary condition, may give rise to the desired deviation in the mixing angles. The mixing angle, θ_{12} is controlled efficiently with $\mu - \tau$ symmetry. We want to preserve this important property even though contribution from charged lepton sector is considered.

7.1 The charged lepton mixing matrix

In the absence of any CP phases, the charged lepton mixing matrix takes the form of a general 3×3 orthogonal matrix. In order to parametrize a 3×3 orthogonal matrix we require three rotational matrices of the following form.

$$R_{12}(\theta) = \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (41)$$

$$R_{23}(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & s_\sigma \\ 0 & -s_\sigma & c_\sigma \end{bmatrix}, \quad (42)$$

$$R_{23}(\tau) = \begin{bmatrix} c_\tau & 0 & s_\tau \\ 0 & 1 & 0 \\ -s_\tau & 0 & c_\tau \end{bmatrix}, \quad (43)$$

where, $s_\omega = \sin \omega$ and $c_\omega = \cos \omega$. We experience nine independent choices of combining these independent rotational matrices in order to generate the general orthogonal matrix [61]. Out of all these choices, we prefer $R = R_{12}(\theta)R_{31}(\tau)R_{23}(\sigma)$ in the charged lepton sector, which is different from the standard parametrization scheme. Again keeping in mind the fact that $R_{ij}^{-1}(\omega)$ plays an equivalent role as $R_{ij}(\omega)$ [61] in the construction of the general orthogonal matrix, we parametrize the charged lepton mixing matrix, U_{PMNS} (see Eq.(40)),

$$\tilde{U}_{eL} = \tilde{R}_{12}^{-1}(\theta)\tilde{R}_{31}^{-1}(\tau)\tilde{R}_{23}(\sigma), \quad (44)$$

and along with the small angle approximation: $s_\omega = \omega$ and $c_\omega = 1 - \omega^2/2$, we finally construct the PMNS matrix, of which the three important elements are,

$$U_{e2} \approx s_{12}^\nu + \frac{c_{12}^\nu}{\sqrt{2}}(\theta - \sigma) - \frac{s_{12}^\nu}{2}(\theta^2 + \sigma^2), \quad (45)$$

$$U_{e3} \approx \frac{1}{\sqrt{2}}(\theta + \sigma), \quad (46)$$

$$U_{\mu 3} \approx \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\tau - \frac{1}{2\sqrt{2}}(\theta^2 + \sigma^2), \quad (47)$$

where $s_{12}^\nu = \sqrt{2}\eta$ and $c_{12}^\nu = \sqrt{1 - 2\eta^2}$. The choice of the σ , θ and τ are arbitrary. So that $\sin \theta_{12}$ as obtained from $M_{\mu\tau}$ is not disturbed, the middle term in the expression of U_{e2} must vanish, $\theta - \sigma = 0$. We choose, $\theta, \sigma = \lambda/2$, ($\lambda = 0.2253 \pm 0.0007$, standard Wolfenstein

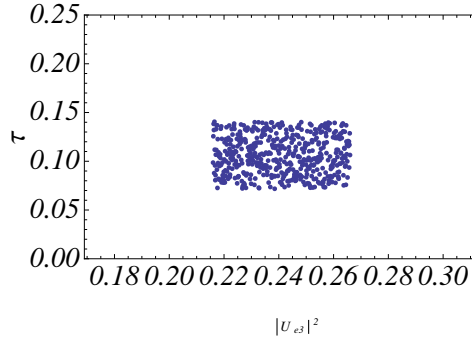


Figure 4: Graphical analysis to fix the parameter, τ against the 1σ range of $\sin^2 \theta_{13} = |U_{e3}|^2$.

parameter [62]) and get $\sin \theta_{13} = |U_{e3}| = \lambda/\sqrt{2}$ [63]. Once, θ and σ are fixed, the choice of τ is guided by the requirement of necessary deviation of θ_{23} from the maximal condition. We see, in Fig (4). with respect to 1σ range of $|U_{e3}|^2$, τ centers around $\tau \sim 0.1 \sim \lambda/2$. Finally we model

$$\tilde{U}_{eL} = \tilde{R}_{12}^{-1}(\lambda/2)\tilde{R}_{31}^{-1}(\lambda/2)\tilde{R}_{23}(\lambda/2). \quad (48)$$

so that,

$$\tilde{U}_{eL}^\dagger \approx \begin{bmatrix} 1 - \frac{\lambda^2}{4} & \frac{\lambda}{2} & \frac{\lambda}{2} \\ -\frac{\lambda}{2} + \frac{\lambda^2}{4} & 1 - \frac{\lambda^2}{4} & -\frac{\lambda}{2} \\ -\frac{\lambda}{2} - \frac{\lambda^2}{4} & \frac{\lambda}{2} - \frac{\lambda^2}{4} & 1 - \frac{\lambda^2}{4} \end{bmatrix} + \mathcal{O}(\lambda^3). \quad (49)$$

7.2 Breaking the $\mu - \tau$ interchange symmetry

Once, the charged lepton contributions are taken into consideration the $\mu - \tau$ symmetry will be perturbed. Finally, we obtain, the corrected neutrino mass matrix, $m_{LL}^\nu(\alpha, \eta, \lambda) = \tilde{U}_{eL}^\dagger M_{\mu\tau} \tilde{U}_{eL}$. The invariant building blocks $I_{i=0,1,2,3}$ (see Table.(1)) of $M_{\mu\tau}$ will now change to,

$$\begin{aligned} I_{i=0,\dots,3}^\lambda &= \tilde{U}_{eL}^\dagger \cdot I_{i=0,\dots,3} \cdot \tilde{U}_{eL} \\ &= I_{i=0,\dots,3} + \Delta I_{i=0,\dots,3}^\lambda + \mathcal{O}(\lambda^3). \end{aligned} \quad (50)$$

The matrices ΔI_i^λ s are listed in Table.(5). We consider the case of $M_{\mu\tau}$ with two free parameters

ΔI_i^λ	
ΔI_0^λ	$\approx \frac{1}{2}\lambda \begin{bmatrix} \lambda & 1 - \frac{1}{2}\lambda & 1 + \frac{1}{2}\lambda \\ 1 - \frac{1}{2}\lambda & -1 - \frac{1}{4}\lambda & -\lambda \\ 1 + \frac{1}{2}\lambda & -\lambda & 1 - \frac{3}{4}\lambda \end{bmatrix}$
ΔI_1^λ	$= -\Delta I_0^\lambda$
ΔI_2^λ	$\approx \frac{1}{2}\lambda \begin{bmatrix} \lambda & -\frac{1}{2}\lambda & 1 + \frac{1}{2}\lambda \\ -\frac{1}{2}\lambda & 1 - \frac{3}{4}\lambda & 0 \\ -\frac{1}{2}\lambda & 0 & -1 - \frac{1}{4}\lambda \end{bmatrix}$
ΔI_3^λ	$\approx \frac{1}{2}\lambda \begin{bmatrix} 0 & -1 + \lambda & -1 - \frac{1}{2}\lambda \\ -1 + \lambda & 2 & 2\lambda \\ -1 - \frac{1}{2}\lambda & 2\lambda & -2 \end{bmatrix}$
$m_{LL}^\nu(\alpha, \eta, \lambda)$	
QDNH- <i>IA</i> : $(I_0 + \Delta I_0^\lambda) - (2\eta - \frac{\alpha}{2})(I_1 - \Delta I_0^\lambda) + 2\alpha(\eta^2 - \frac{1}{4})(I_2 + \Delta I_2^\lambda) + \alpha\eta(1 - 2\eta^2)^{1/2}(I_3 + \Delta I_3^\lambda)$	
QDNH- <i>IB</i> : $(I_0 + \Delta I_0^\lambda) + (2\eta - \frac{\alpha}{2})(I_1 - \Delta I_0^\lambda) - 2\alpha(\eta^2 - \frac{1}{4})(I_2 + \Delta I_2^\lambda) - \alpha\eta(1 - 2\eta^2)^{1/2}(I_3 + \Delta I_3^\lambda)$	
QDNH- <i>IC</i> : $-(I_0 + \Delta I_0^\lambda) + (2\eta - \frac{\alpha}{2})(I_1 - \Delta I_0^\lambda) - 2\alpha(\eta^2 - \frac{1}{4})(I_2 + \Delta I_2^\lambda) - \alpha\eta(1 - 2\eta^2)^{1/2}(I_3 + \Delta I_3^\lambda)$	
QDIH- <i>IA</i> : $2\eta(I_0 + \Delta I_0^\lambda) - (1 - \frac{\alpha}{2})(I_1 - \Delta I_0^\lambda) + 2\alpha(\eta^2 - \frac{1}{4})(I_2 + \Delta I_2^\lambda) + \alpha\eta(1 - 2\eta^2)^{1/2}(I_3 + \Delta I_3^\lambda)$	
QDIH- <i>IB</i> : $2\eta(I_0 + \Delta I_0^\lambda) + (1 - \frac{\alpha}{2})(I_1 - \Delta I_0^\lambda) - 2\alpha(\eta^2 - \frac{1}{4})(I_2 + \Delta I_2^\lambda) - \alpha\eta(1 - 2\eta^2)^{1/2}(I_3 + \Delta I_3^\lambda)$	
QDIH- <i>IC</i> : $-2\eta(I_0 + \Delta I_0^\lambda) + (1 - \frac{\alpha}{2})(I_1 - \Delta I_0^\lambda) - 2\alpha(\eta^2 - \frac{1}{4})(I_2 + \Delta I_2^\lambda) - \alpha\eta(1 - 2\eta^2)^{1/2}(I_3 + \Delta I_3^\lambda)$	
$U_{PMNS} \approx \begin{bmatrix} c_{12}^\nu(1 - \frac{1}{4}\lambda^2) & s_{12}^\nu(1 - \frac{1}{4}\lambda^2) & \frac{\lambda}{\sqrt{2}} \\ -\frac{s_{12}^\nu}{\sqrt{2}} - \frac{\lambda}{2}(1 - \frac{\lambda}{2})(\frac{s_{12}^\nu}{\sqrt{2}} + c_{12}^\nu) & \frac{c_{12}^\nu}{\sqrt{2}} + \frac{\lambda}{2}(1 - \frac{\lambda}{2})(\frac{c_{12}^\nu}{\sqrt{2}} - s_{12}^\nu) & \frac{1}{\sqrt{2}} - \frac{\lambda}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}}(1 - \frac{\lambda}{2})s_{12}^\nu - \frac{\lambda}{2}(1 + \frac{\lambda}{2})c_{12}^\nu & -\frac{1}{\sqrt{2}}(1 - \frac{\lambda}{2})c_{12}^\nu - \frac{\lambda}{2}(1 + \frac{\lambda}{2})s_{12}^\nu & \frac{1}{\sqrt{2}} + \frac{\lambda}{2\sqrt{2}} \end{bmatrix},$ $s_{12}^\nu = \sqrt{2}\eta, \quad c_{12}^\nu = (1 - 2\eta^2)^{1/2}$	

Table 5: The perturbation to the respective building block matrices, I_i s are estimated in terms of ΔI_i s. The corresponding textures of the corrected mass matrices $m_{LL}^\nu(\alpha, \eta, \lambda)$ are also described. The lepton mixing matrix which is now modified from $U_{\nu L}$ to $U_{eL}^\dagger U_{\nu L}$ is also presented.

(α, η) for QDNH-*Type A* case (see Eq.(32)), as example.

$$\begin{aligned}
m_{LL}^\nu(\alpha, \eta, \lambda) &= \tilde{U}_{eL}^\dagger \cdot M_{\mu\tau}(\alpha, \eta) \tilde{U}_{eL} \\
&= I_0^\lambda - (2\eta - \frac{\alpha}{2})I_1^\lambda + 2\alpha(\eta^2 - \frac{1}{4})I_2^\lambda + \alpha\eta(1 - 2\eta^2)^{1/2}I_3^\lambda, \\
&= \begin{bmatrix} \alpha - 2\eta - 2\alpha\eta^2 & -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} \\ -\alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} - \eta + \alpha\eta^2 & \frac{1}{2} + \eta - \alpha\eta^2 \\ \alpha\eta(1 - 2\eta^2)^{\frac{1}{2}} & \frac{1}{2} + \eta - \alpha\eta^2 & \frac{1}{2} - \eta + \alpha\eta^2 \end{bmatrix} \\
&\quad + \frac{\lambda}{2} \left(1 - 2\eta - \frac{\alpha}{2}\right) \begin{bmatrix} \lambda & 1 - \frac{1}{2}\lambda & 1 + \frac{1}{2}\lambda \\ 1 - \frac{1}{2}\lambda & -1 - \frac{1}{4}\lambda & -\lambda \\ 1 + \frac{1}{2}\lambda & -\lambda & 1 - \frac{3}{4}\lambda \end{bmatrix} \\
&\quad + \alpha\lambda \left(\eta^2 - \frac{1}{4}\right) \begin{bmatrix} \lambda & -\frac{\lambda}{2} & 1 + \frac{1}{2}\lambda \\ -\frac{\lambda}{2} & 1 - \frac{3}{4}\lambda & 0 \\ 1 + \frac{1}{2}\lambda & 0 & -1 - \frac{1}{4}\lambda \end{bmatrix} \\
&\quad + \frac{\alpha\eta\lambda}{2}(1 - 2\eta^2)^{1/2} \begin{bmatrix} 0 & -1 + \lambda & -1 - \frac{1}{2}\lambda \\ -1 + \lambda & 2 & 2\lambda \\ -1 - \frac{1}{2}\lambda & 2\lambda & -2 \end{bmatrix} + \mathcal{O}(\lambda^3)
\end{aligned} \tag{51}$$

The details of the texture for other QDN cases are described in Table.(5). The texture of the PMNS matrix, $U_{PMNS} = U_{eL}^\dagger \cdot U_{\nu L}$, is presented in Table.(5). We obtain,

$$\sin^2 \theta_{12} = 2\eta^2 + \mathcal{O}(\lambda^3), \tag{52}$$

$$\sin^2 \theta_{13} = \frac{1}{2}\lambda^2 + \mathcal{O}(\lambda^3), \tag{53}$$

$$\sin^2 \theta_{23} = \frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{8}\lambda^2 + \mathcal{O}(\lambda^3). \tag{54}$$

8 Numerical calculation

We assign certain ranges to the free parameter α and η respectively. Based on the 1σ range of the physical observable quantities available from Global data analysis [45], we assign $\alpha = 1.5939 - 1.6239$ (QDNH-IA), $0.0080 - 0.0220$ (QDNH-IB,IC), $1.9945 - 1.9948$ (QDIH-IA), $0.0052 - 0.0055$ (QDIH-IB,IC), and $\eta = 0.3814 - 0.4031$. The input parameter $m_0 \sim 0.08 \text{ eV}$. and $\lambda = 0.2253$. We have now four parameters, out of which α and η are free and the number of unknowns present is six.

8.1 Observable parameters in oscillation experiments and cosmological observation

We apply the six QDN neutrino mass matrices $m_{LL}^\nu(\alpha, \eta, \lambda)$ to study their relevance in the oscillation experiments. It is found that under a suitable choice of the free parameters (α, η) , all the six QDN models are equally capable of describing both TBM and TBM-deviated scenarios (see Table.(6)) and are indistinguishable. QDNH model says, $|m_1|, |m_2| \sim 0.06 \text{ eV}$, $|m_3| \sim 0.08 \text{ eV}$, while $|m_2|, |m_3| \sim 0.08 \text{ eV}$, $m_1 \sim 0.06 \text{ eV}$ for QDIH case. For both the cases, $\Delta m_{21}^2 \sim 7.6 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 \sim 2.4 \times 10^{-3} \text{ eV}^2$. The mixing angle parameters are $\sin^2 \theta_{13} \simeq 0.025$, $\sin^2 \theta_{12} \simeq 0.32$ and $\sin^2 \theta_{23} \simeq 0.39$. Also $\Sigma|m_i| \simeq 0.21 \text{ eV}$ (QDNH case) and $\Sigma|m_i| \simeq 0.23 \text{ eV}$ (QDIH case).

QD	NH-IA	NH-IB	NH-IC	IH-IA	IH-IB	IH-IC
α (TBM)	1.626	0.0068	0.0068	1.9946	0.0054	0.0054
α	1.5929	0.0071	0.0071	1.9946	0.0054	0.0054
η (TBM)	0.4083	0.4083	0.4083	0.4083	0.4083	0.4083
η	0.40	0.40	0.40	0.3987	0.3987	0.3987
m_0 eV	0.082	0.082	0.082	0.084	0.084	0.084
m_1 eV(TBM)	0.06638	0.06639	0.06639	0.08355	0.08355	0.08355
m_2 eV(TBM)	- 0.06695	0.06695	0.06695	-0.084	0.084	0.084
m_3 eV(TBM)	0.082	0.082	-0.082	0.06859	0.06859	-0.06859
m_1 eV	0.06502	0.06502	0.06502	0.08355	0.08355	0.08355
m_2 eV	-0.0656	0.0656	0.0656	-0.084	0.084	0.084
m_3 eV	0.082	0.082	-0.082	0.0672	0.0672	-0.0672
$\Delta m_{21}^2(10^{-5}eV^2)$ (TBM)	7.645	7.435	7.435	7.60	7.60	7.60
$\Delta m_{31}^2(10^{-3}eV^2)$ (TBM)	2.318	2.316	2.316	-2.352	-2.28	-2.28
$\Delta m_{21}^2(10^{-5}eV^2)$	7.605	7.605	7.605	7.60	7.60	7.60
$\Delta m_{21}^2(10^{-3}eV^2)$	2.497	2.497	2.497	-2.464	-2.464	-2.464
Σm_i eV (TBM)	0.2153	0.2154	0.2154	0.23613	0.23613	0.23613
Σm_i eV	0.21262	0.21262	0.21262	0.23475	0.23475	0.23475
$\sin^2 \theta_{12}$	0.319	0.319	0.319	0.3195	0.3195	0.3195
$\sin^2 \theta_{13}$	0.0252	0.0252	0.0252	0.0252	0.0252	0.0252
$\sin^2 \theta_{23}$	0.3943	0.3943	0.3943	0.3943	0.3943	0.3943
m_{ν_e} eV	0.06582	0.06582	0.06582	0.0835	0.0835	0.0835
m_{ee} eV	0.02452	0.06590	0.06174	0.03063	0.083625	0.08021

Table 6: The study of the six cases of Quasi degenerate neutrino mass model for both TBM mixing and deviation from TBM mixing. The analysis is done with the parameters (α, η, λ) and input m_0 . m_0 is fixed at 0.082 eV (QDNH) and 0.084 eV (QDIH) respectively. The free parameter α is related with absolute masses. The free parameter η controls both masses and the solar angle. $\lambda = 0.2253$, the Wolfenstein parameter is related with deviation of reactor angle from zero and that for atmospheric from maximal condition.

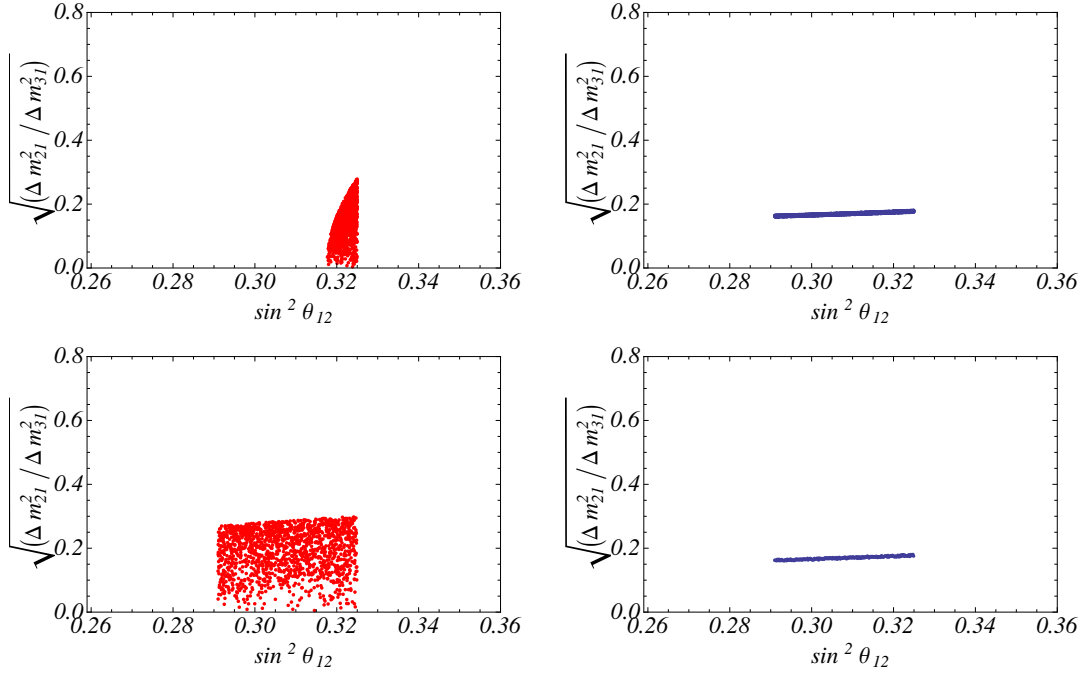


Figure 5: The correlation plots in the plane of $\sqrt{\Delta m_{21}^2 / \Delta m_{31}^2}$ and $\sin^2 \theta_{sol}$ for different cases of QDNH-IA (top-left), QDNH-IB,IC (bottom-left), QDIH-IA (top-right) and QDIH-IB,IC (bottom-right). The bounds on $\sqrt{\Delta m_{21}^2 / \Delta m_{31}^2}$ are found to be sharp for QDIH cases. The experimental value of this quantity must lie close to 0.2. For QDNH-IA case, we obtain a bound on $\sin^2 \theta_{sol}$ around a value of 0.32.

We study a quantity $\sqrt{\Delta m_{21}^2} / \sqrt{\Delta m_{31}^2}$, which according to the global data analysis lies near to 0.2. The correlation plots in the plane $\sqrt{\Delta m_{21}^2} / \sqrt{\Delta m_{31}^2}$ and $\sin^2 \theta_{12}$ for all QDN models are shown in Fig (4). We see for QDNH-Type IA case, there exists a sharp bound on $\sin^2 \theta_{12}$ around 0.32 which is the experimental best-fit of $\sin^2 \theta_{12}$ according to Global data analysis [44].

8.2 Absolute electron neutrino mass (m_{ν_e}) and Effective Majorana neutrino mass (m_{ee})

Besides the oscillation experiments and the cosmological bound on $\Sigma |m_i|$, There are other two important quantities : effective electron neutrino mass, m_{ν_e} appearing in β -decay and effective Majorana mass m_{ee} , appearing in neutrino-less double β -decay experiment and are useful for the study of nature of the neutrino masses.

$$m_{\nu_e} = (\Sigma m_i^2 |U_{ei}|^2)^{1/2}, \quad (55)$$

$$m_{ee} = |m_1 |U_{e1}|^2 + m_2 |U_{e2}|^2 + m_3 |U_{e3}|^2|. \quad (56)$$

The results of Mainz [64] and Toitsk [65] Tritium β -decay experiments, we obtain, $m_{\nu_e} < 2.2 \text{ eV}$. The upcoming KATRIN experiment [66], expects the sensitivity upto $m_{\nu_e} \sim 0.3 \text{ eV}$. In the present work, the QDNH and QDIH models predict, $m_{\nu_e} \sim 0.07 \text{ eV}$ and $m_{\nu_e} \sim 0.08 \text{ eV}$ respectively.

The HM group [67–69] and IGEX [70–72] groups reported the upper limit of m_{ee} to $0.3 - 1.3 \text{ eV}$. The CUORICINO [73] experiment gives an improved upper bound on m_{ee} , $m_{ee} < 0.23 - 0.85 \text{ eV}$. This is still considered somewhat controversial [72, 74], and requires independent confirmation. The experiments such as CUORE [75, 76], GERDA [77], NEMO [78–80] and Majorana [81, 82] will attempt to improve the sensitivity of the measurement down to about $m_{ee} \simeq (0.05 - 0.09) \text{ eV}$. Hence in that respect the QDN models are of immense importance. In

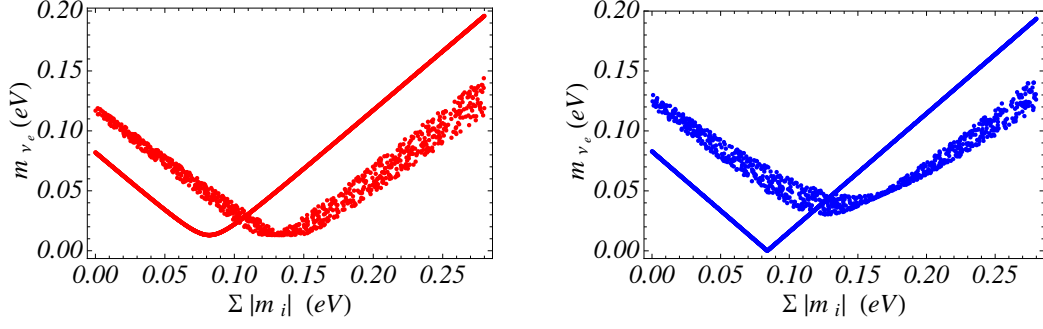


Figure 6: A study of the correlation in the plane of m_{ν_e} and Σm_i . Left: QDNH case, Right: QDIH case.

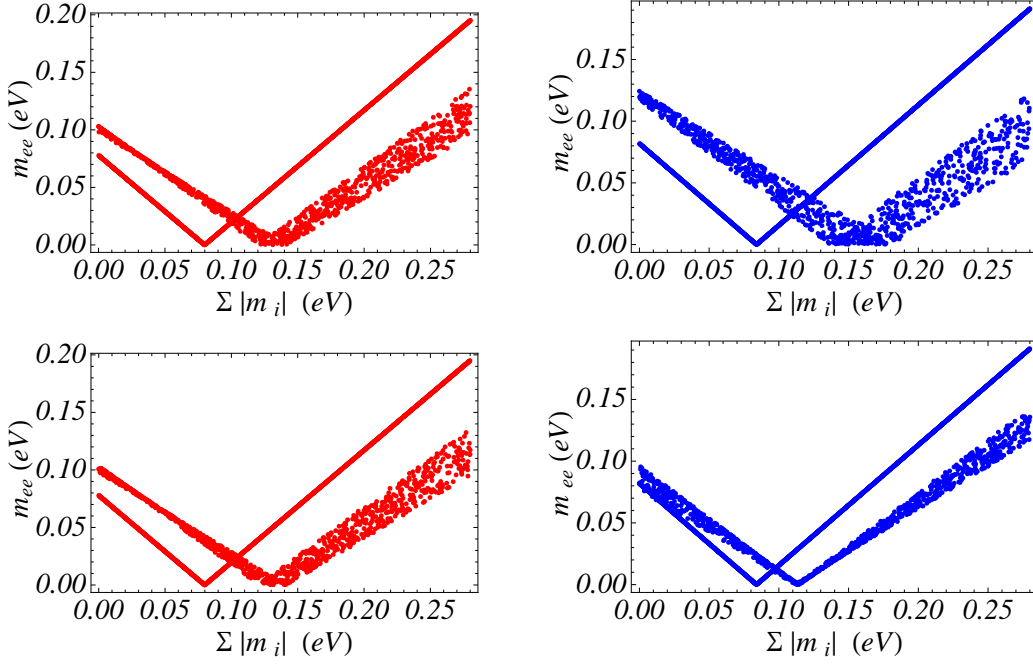


Figure 7: A study of the correlation in the plane of m_{ee} and Σm_i . Top-left: QDNH-Type IA case; top-right: QDNH-Type IB, IC cases; bottom-left: QDIH-Type IA case; bottom-right: QDIH-Type IB, IC cases.

our present work, QDNH and QDIH-both Type IB and Type IC models predict $m_{ee} \simeq 0.06 \text{ eV}$ and $m_{ee} \simeq 0.08 \text{ eV}$ respectively. The predictions given by Type IA cases of both QDNH and QDIH models are interesting in the sense that they leave a scope for the future experiments to go down upto a sensitivity of $m_{ee} \simeq 0.02 \text{ eV}$ and $m_{ee} \simeq 0.03 \text{ eV}$ respectively. The correlation plots are studied in the plane m_{ν_e} (and m_{ee}) and $\Sigma |m_i|$ (Fig (6) and Fig (7)).

9 Discussion: How to discriminate different QDN models?

We have tried to bring all the eleven cases involving six QDN, three NH and two IH cases under same roof of parametrization by introducing four common independent building block matrices, $I_{i=0,1,2,3}$. The idea of fragmentation is guided by the quest of some mechanism to save the internal texture of $M_{\mu\tau}$ against the changing solar angle. The I_i s when incorporated with the free parameters in a proper way, lead to an important feature of $M_{\mu\tau}$, $\sin \theta_{12} = \sqrt{2}\eta$. θ_{12} is

expressible in terms single parameter only, unlike the general $M_{\mu\tau} = M_{\mu\tau}(x, y, z, w)$ where θ_{12} requires the knowledge of four free parameters (x, y, z, w) (see Eq.(8)). This is also interesting to note that one of the building blocks, I_3 has got the same eigenstates as predicted by BM mixing (see Table.(1)). The existence of this invariant textures within the mass matrix seems to be relevant and we hope that a fruitful investigation is subjected to the study of underlying discrete flavour symmetry groups.

Charged lepton correction is considered as a significant tool in order to break the $\mu - \tau$ symmetry [28–34]. The models where θ_{13}^ν is very small, contributions to θ_{13} is implemented mostly from charged lepton sector. Also, this tool is very important for those models where $\theta_{23} = \pi/4$ and it may provide consistency with the LMA MSW solutions [83]. In GUT scenarios also, one finds in addition to the breaking of $\mu - \tau$ symmetry in the neutrino sector, charged lepton corrections are unavoidable [84, 85]. Regarding the parametrization of U_{eL} , we have followed a parametrization scheme different from that of standard one. This step is motivated by the fact that a particular choice of parametrization does not affect the final observables, but a suitable choice can make the mathematics easier. The parametrization of U_{eL} respects the GUT motivated, new QLC relation, $\theta_{13} \sim \theta_c/\sqrt{2}$ [63]. In our parametrization U_{eL} does not affect the prediction of θ_{12} from neutrino sector.

Our work started with the following motivations. They are,

- (1) Whether QDN neutrino mass models are equally possible like that of NH and IH models?
- (2) How to discriminate the QDN models?

In the background of the oscillation experiments, we have tried to answer to the first question by testing the efficiency of each $m_{LL}^\nu(\alpha, \eta, \lambda)$ in predicting the values of five observational parameters and comparing those with Global data. In that context we find the existence of QDN neutrinos of both NH and IH pattern is undisputed. Above all, all the six QDN models sound equally possible (see Table.(6)). Hence only the oscillation experiments are not sufficient enough to answer to the second question. But here we want to mention that QDNH-Type A model shows a strong preference for $\sin^2 \theta_{12} = 0.32$, which is the best-fit result according to Global analysis done by Forero *et.al* [44], evident from the correlation plot in Fig(5).

We have tried to find out the answer of the second question in the framework of β -decay and $0\nu\beta\beta$ -decay experiments. But all the six QDN models predict the quantities m_{ν_e} and m_{ee} , below the upper bounds of the past experiments and interestingly they are much closer to the sensitivities expected to be achieved in the future experiments. In our analysis QDNH-Type-A model leaves a scope for future experiments to go down upto $m_{ee} \sim 0.02 \text{ eV}$.

In section (4), it has been shown that QDN nature of neutrinos permits the mass scale, $0.05 \leq m_0 \leq 0.1 \text{ eV}$. But, concerning a fair degree of degeneracy, the range is modified to, $0.07 \leq m_0 \leq 0.1 \text{ eV}$. The ansatz regarding the correlation, $\beta : \eta \simeq 2$ plays an important role in the transition from $M_{\mu\tau}(\alpha, \beta, \eta) \rightarrow M_{\mu\tau}(\alpha, \eta)$. This ansatz holds good for the mass scale $m_0 \sim 0.07 - 0.09 \text{ eV}$, over the 3σ range of β and η . If there are three free parameters (α, β, η) present in m_{LL}^ν , degrees of arbitrariness is also quite higher. Although this ansatz restricts the arbitrariness of θ_{12} to some extent, yet only two free parameters α and η (with $\lambda = 0.2253$ and input m_0 fixed at 0.08 eV) are sufficient to predict five observational parameters (related with oscillation experiments), in close agreement with that of experimental 1σ range of data. The parametrization respects both TBM and small TBM-deviated cases. In this context the ansatz $\beta = 2\eta$, appears to be relevant and natural.

We hope that perhaps the cosmological upper bound on Σm_i have some relevance with the discrimination of the six models. So long we adhere to $\Sigma m_i < 0.28 \text{ eV}$ [19], both QDNH and QDIH models are safe which predict, $\Sigma m_i = 0.212$ and $\Sigma m_i = 0.235$ respectively for input mass scale $m_0 \sim 0.08 \text{ eV}$. But the recent analysis supports for a tighter upper bound, $\Sigma m_i < 0.23 \text{ eV}$ [86]. If so, the QDIH model seems to be insecure in our analysis. If we believe the ansatz, $\beta : \eta = 2$ to be natural, then with lowering the mass scale from 0.08 eV , and controlling

α and η , we can achieve $\Sigma m_i < 0.23 \text{ eV}$ for QDIH case, and also this will favor the TBM deviated condition. But at the same time, it will give rise to a serious problem that QDIH model with $m_0 < 0.08 \text{ eV}$ will completely discard $\theta = \sin^{-1}(1/\sqrt{3})$ (TBM), because corresponding to that angle, Δm_{31}^2 will be outside the 3σ range. But the solar angle $\theta = \sin^{-1}(1/\sqrt{3})$, is still relevant within 1σ [44] or 2σ [45] range. So on this basis can we discard QDIH model? But we hope it will be too hurry to come to any conclusion. There is a possibility that by assuming $\beta : \eta = c$, where $c \neq 2$ but $c \sim 2$ (which is allowed indeed), and then lowering of m_0 , may solve this problem and can make QDIH models safe.

The discussion so far tells that on phenomenological ground, there is no dispute on the existence of quasi-degenerate neutrinos with $m_i < 0.1 \text{ eV}$, in nature. But the question whether it is of NH type or IH type, is still not clear. In this regard, we expect that possible answer may emerge from the observed Baryon asymmetry of the universe ($\eta_B = 6.5_{-0.5}^{+0.4} \times 10^{-10}$) [12–14, 39]. The calculation of η_B requires the texture of heavy right handed Majorana neutrino mass matrix, M_{RR} . With a suitable choice of Dirac neutrino mass matrix, m_{LR} allowed by $SO(10)$ GUT, we can transit from m_{LL}^ν (parametrized so far) to M_{RR} by employing the inversion of Type I see-saw formula, $M_{RR} = -m_{LR}^T m_{LL}^{\nu-1} m_{LR}$. We hope that significant physical insight can be fetched from this approach and it would be possible to figure out the most favorable QDN models out of the six. Unlike, in Refs. [12, 13], the parametrization of mass matrices involves only two free parameters (α, η) and no other input constant terms which are also different for TBM and TBM deviated scenarios. The prediction of θ_{12} involves (η, ϵ, c, d) , whereas in our parametrization it depends on η only. With minimum number of parameters, we have achieved a better control over mass matrices. In contrast to Refs. [12, 13], with our parametrization of m_{LL}^ν , certain analytical structure of $U_{\nu L}$ is also possible. We hope, this parametrization will be useful for other phenomenological studies also.

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